Single impurity operators at critical wrapping order in the β-deformed $\mathcal{N}$ = 4 SYM

JHEP08(2009)034

(http://iopscience.iop.org/1126-6708/2009/08/034)

JHEP

Published by IOP Publishing for SISSA

Received: *May 17, 2009*
Accepted: *July 17, 2009*
Published: *August 10, 2009*

Single impurity operators at critical wrapping order in the β-deformed $\mathcal{N} = 4$ SYM

F. Fiamberti,[a,b] **A. Santambrogio,**[b] **C. Sieg**[c] **and D. Zanon**[a,b]

[a] *Dipartimento di Fisica, Università degli Studi di Milano, Via Celoria 16, 20133 Milano, Italy*
[b] *INFN — Sezione di Milano, Via Celoria 16, 20133 Milano, Italy*
[c] *The Niels Bohr International Academy, The Niels Bohr Institute, Blegdamsvej 17, DK-2100 Copenhagen, Denmark*
E-mail: francesco.fiamberti@mi.infn.it, alberto.santambrogio@mi.infn.it, csieg@nbi.dk, daniela.zanon@mi.infn.it

Abstract: We study the spectrum of one single magnon in the superconformal β-deformed $\mathcal{N} = 4$ SYM theory in the planar limit. We compute the anomalous dimensions of one-impurity operators $\mathcal{O}_{1,L} = \mathrm{tr}(\phi Z^{L-1})$, including wrapping contributions at their critical order L.

Keywords: Supersymmetric gauge theory, Superspaces, AdS-CFT Correspondence, Integrable Field Theories

ArXiv ePrint: 0811.4594

doi:10.1088/1126-6708/2009/08/034

JHEP08(2009)034

Contents

JHEP08(2009)034

1 Introduction

In some recent papers [1–3] we have studied finite size effects in the anomalous dimensions of gauge invariant composite operators. More precisely we have addressed this issue by performing the analysis in two special settings, i.e. in the planar limit of the superconformal $\mathcal{N} = 4$ SYM theory and in the exactly marginal deformation of $\mathcal{N} = 4$ SYM theory preserving $\mathcal{N} = 1$ supersymmetry.

The interest in this kind of calculation stems from the quest for an ever deeper understanding of the AdS/CFT correspondence [4] and the comparison of the spectra of the gauge theory and the string theory. On the gauge theory side important progress has been achieved thanks to the realization that some sectors of the theory can be described in terms of quantum spin chains for which in many cases a Hamiltonian, an asymptotic Bethe ansatz [5–9] and even an exact, factorized S-matrix corrected by a dressing phase [10–13] are available. These results were primarily obtained for the anomalous dimensions of operators of infinite length whose analysis is considerably simplified.

When finite size effects become important new interactions need to be taken into account. On the string theory side recent papers have studied finite size contributions in the spectrum of magnons [14–23]. On the field theory side wrapping interactions [9] have been analyzed [24] in the $\mathcal{N} = 4$ SYM theory. The anomalous dimension of the composite operator $\mathrm{tr}(\phi[Z, \phi]Z)$ has been computed at four loops in [1, 2, 25, 26].

In [3] we have considered finite size effects in a less symmetric setting. More precisely we have focused on the β-deformed $\mathcal{N} = 4$ SYM theory obtained from the original $\mathcal{N} = 4$ theory by modifying the superpotential via the substitution

$$ig\ \mathrm{tr}\,(\phi\,\psi\,Z - \phi\,Z\,\psi) \quad \longrightarrow \quad ih\ \mathrm{tr}\left(\mathrm{e}^{i\pi\beta}\,\phi\,\psi\,Z - \mathrm{e}^{-i\pi\beta}\,\phi\,Z\,\psi\right) \tag{1.1}$$

with h and β complex constants. In [27] it was argued that this theory becomes conformally invariant if the constants h and β satisfy one condition. Indeed in [28] it has been shown that for a real deformation parameter β this deformed $\mathcal{N}=1$ theory becomes superconformal, in the planar limit to all perturbative orders, if

$$h\bar{h} = g^2 \ . \tag{1.2}$$

The AdS/CFT correspondence predicts that this deformed theory is equivalent to the Lunin-Maldacena string theory background [29]. The existence of integrable structures in the deformed string background has been analyzed in [30–32]. Finite size effects of single magnons were discussed in [33].

A new feature of the deformed theory as compared to the situation in $\mathcal{N}=4$ SYM is given by the fact that one-impurity states are not protected by supersymmetry. In [3] we have computed the anomalous dimensions of one-impurity and two-impurity operators up to four-loop order and made some partial calculations and conjectures for the simplest single-impurity operator $\mathcal{O}_{1,L} = \mathrm{tr}(\phi Z^{L-1})$ at higher order L in perturbation theory.

This paper is the natural extension of the work presented in [3]. Here we want to compute in the planar limit the anomalous dimensions of the one-impurity operators $\mathcal{O}_{1,L} = \mathrm{tr}(\phi Z^{L-1})$, including wrapping contributions at their critical order L.

The anomalous dimension of a composite operator $\mathcal{O}$ can be obtained from the divergent diagrams that contribute to its one-point function. In dimensional regularization, if the operator is renormalized multiplicatively, it is simply given by

$$\gamma(\mathcal{O}) = \lim_{\varepsilon\to 0}\left[\varepsilon g \frac{\mathrm{d}}{\mathrm{d}g}\log \mathcal{Z}_{\mathcal{O}}(g,\varepsilon)\right] , \tag{1.3}$$

where

$$\mathcal{O}_{\mathrm{ren}} = \mathcal{Z}_{\mathcal{O}}\mathcal{O}_{\mathrm{bare}} \tag{1.4}$$

and ε is the dimensional regularization parameter. Thus we want to compute up to L loops all the divergent contributions to the one-point function of the operator $\mathcal{O}_{1,L}$ and isolate the $1/\varepsilon$ poles. At first sight this program looks very ambitious and complicated. In fact the use in conjunction of integrability properties and of superspace techniques proved so powerful that we were able to reduce and structure the whole calculation into a manageable form. The computation is organized through the sequence of the following steps:
First we obtain the contributions to the anomalous dimension which do not contain wrapping interactions, up to perturbative order L. This we achieve by starting from the knowledge of the anomalous dimensions of long single-impurity states of the SU(2) sector. If the corresponding operator $\mathcal{O}_{\mathrm{as}}$ has such a length that wrapping interactions do not contribute, the anomalous dimension at a given perturbative order can be obtained from the all-loop result [28]

$$\gamma(\mathcal{O}_{\mathrm{as}}) = -1 + \sqrt{1+4\lambda\left|q-\frac{1}{q}\right|^2} = -1 + \sqrt{1+16\lambda\sin^2(\pi\beta)}\,, \tag{1.5}$$

where $\lambda = \frac{g^2 N}{16\pi^2}$. This result is correct only in the asymptotic regime.

JHEP08(2009)034

Thus in order to obtain the anomalous dimensions for $\mathcal{O}_{1,L}$ up to order L, we need to subtract the range-$(L+1)$ contributions from the L-order expansion of (1.5) and finally compute explicitly the wrapping diagrams at the critical order L.

The number of wrapping graphs that one has to consider at L loops becomes so large that in [3] we thought the calculation could not be performed exactly. In fact a judicious analysis in terms of superfields and supergraphs allows to discover a huge number of cancellations such that in the end only a few restricted classes of supergraphs are shown to be relevant. In the next section we consider these superspace Feynman diagram calculations, showing how the cancellations occur and performing the D-algebra on the relevant contributions. In section 3 we study the integrals. In section 4 we comment on our results. The strategy to explicitly compute the required integrals can be found in the appendix.

2 The power of superspace

We want to compute the anomalous dimension of the composite operator $\mathcal{O}_{1,L} = \mathrm{tr}(\phi Z^{L-1})$, including wrapping contributions up to L loops in the planar limit following the strategy introduced in [1–3]. While doing similar calculations at lower loop order we learned that superspace techniques are very efficient for organizing the work and moreover we found many unexpected cancellations to occur. In this section we will review briefly the general rules of the $\mathcal{N}=1$ superspace formalism, primarily to set our notations, and we will explain the reasons that lead to the many cancellations we had noticed in our previous works. It has been the realization of these general great simplifications that encouraged us in undertaking this project.

The action of $\mathcal{N}=1$ β-deformed SYM is described in terms of one real vector superfield V and three chiral superfields ϕ, ψ, Z that we denote collectively by ϕ^i. Following the notations and conventions of [34] it is given by

$$\begin{aligned} S = &\int \mathrm{d}^4x\, \mathrm{d}^4\theta\ \mathrm{tr}\left(\mathrm{e}^{-gV}\,\bar{\phi}_i\,\mathrm{e}^{\mathrm{g}V}\,\phi^i\right) + \frac{1}{2g^2}\int \mathrm{d}^4x\, \mathrm{d}^2\theta\ \mathrm{tr}\left(W^\alpha W_\alpha\right) \\ &+ ih\int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathrm{tr}\left(\mathrm{e}^{i\pi\beta}\,\phi_1\,\phi_2\,\phi_3 - \mathrm{e}^{-i\pi\beta}\,\phi_1\,\phi_3\,\phi_2\right) + \mathrm{h.c.}\,, \end{aligned} \tag{2.1}$$

where $W_\alpha = \bar{D}^2\left(\mathrm{e}^{-gV}\,D_\alpha\,\mathrm{e}^{\mathrm{g}V}\right)$, and $V = V^a T^a$, $\phi^i = \phi^a_i T^a$, $i=1,2,3$, T^a being matrices satisfying the SU(N) algebra

$$[T_a, T_b] = i f_{abc} T_c \tag{2.2}$$

and normalized as

$$\mathrm{tr}(T_a T_b) = \delta_{ab}\ . \tag{2.3}$$

In order to compute Feynman diagrams we need propagators and vertices that can be easily obtained from the action (2.1). The superfield propagators are given in momentum space by

$$\langle V^a V^b\rangle = -\frac{\delta^{ab}}{p^2}\,, \qquad\qquad \langle \phi^a_i\,\bar{\phi}^b_j\rangle = \delta_{ij}\frac{\delta^{ab}}{p^2}\ . \tag{2.4}$$

The vertices that we need are

$$
\begin{aligned}
V_1 &= \; g f_{abc}\delta^{ij}\bar{\phi}_i^a V^b \phi_j^c\,, \qquad\qquad V_2 = \frac{g^2}{2}\delta^{ij} f_{adm} f_{bcm} V^a V^b \bar{\phi}_i^c \phi_j^d\,,\\
V_3 &= -h f_{abc}(\mathrm{e}^{i\pi\beta}\,\phi_1^a\,\phi_2^b\,\phi_3^c - \mathrm{e}^{-i\pi\beta}\,\phi_1^a\,\phi_3^b\,\phi_2^c)\,,\\
\bar{V}_3 &= -\bar{h} f_{abc}(\mathrm{e}^{-i\pi\beta}\,\bar{\phi}_1^a\bar{\phi}_2^b\bar{\phi}_3^c - \mathrm{e}^{i\pi\beta}\,\bar{\phi}_1^a\bar{\phi}_3^b\bar{\phi}_2^c)\,,
\end{aligned}
\tag{2.5}
$$

with additional $\bar{D}^2$, D^2 factors for each chiral, antichiral line respectively. It is easy to realize that since our operator $\mathcal{O}_{1,L}$ has length L, vertices containing three or more vector superfields V never enter the L-loop calculation.

As anticipated in the introduction essentially all diagrams without wrapping interactions need not be examined since their contribution to the L-loop anomalous dimension of $\mathcal{O}_{1,L}$ is simply obtained from the L-order expansion of the asymptotic result in (1.5)

$$
\gamma_L^{\mathrm{as}} = \alpha_L\,\lambda^L \sin^{2L}(\pi\beta)\,, \qquad \alpha_L = -(-8)^L\frac{(2L-3)!!}{L!}\,. \tag{2.6}
$$

In fact the above result is valid for single-impurity states of length greater than L. Thus we need correct it by subtracting the range-$(L+1)$ contributions. Here is where our Feynman diagram computation really starts. It is organized in two separate steps:

(a) subtraction of the range $L+1$ diagrams

(b) computation of all the L-loop wrapping diagrams.

According to the general procedure one has to consider a given supergraph and complete the D-algebra in order to reduce it to a standard divergent graph. All graphs that give rise to finite integrals are immediately discarded since they are not relevant for the computation of the anomalous dimension of the composite operator. In [2] we have shown that in order to produce divergent contributions the D-algebra must be performed in such a way that no spinor derivative is moved out onto the external lines, except for derivatives on scalar propagators that do not belong to any loop. The following analysis makes substantial use of this very strong result.

Now we start considering the Feynman diagrams that we have to subtract in order to cancel the unwanted range-$(L+1)$ contributions contained in (2.6). As in the lower order examples considered in [3], we have to subtract only one diagram (and its reflection) constructed entirely in terms of chiral interactions. It is shown in figure 1 and denoted by S_L. Introducing the same notations for the chiral structures of the supergraphs as in [3], we find

$$
\begin{aligned}
S_L &\to (g^2N)^L J_L\,[\boldsymbol{\chi}(1,2,\ldots,L) + \boldsymbol{\chi}(L,\ldots,2,1)]\\
&= (g^2N)^L J_L\,(q-\bar{q})^2\left[q^{2(L-1)} + \bar{q}^{2(L-1)}\right]\,,
\end{aligned}
\tag{2.7}
$$

where J_L is given by the L-loop integral in figure 2a. Thus the term to be subtracted from (2.6) is

$$
\delta\gamma_L^{\mathrm{s}} = -2L\lim_{\varepsilon\to 0}(\varepsilon S_L)\,. \tag{2.8}
$$

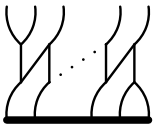
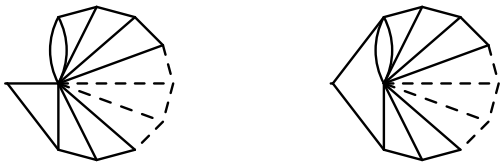


Figure 1. L-loop, range-$(L+1)$ diagram S_L



(a) J_L **(b)** K_L

Figure 2. L-loop integrals from diagrams S_L and $W_{L,0}$

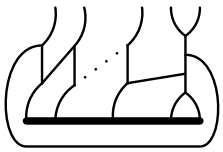
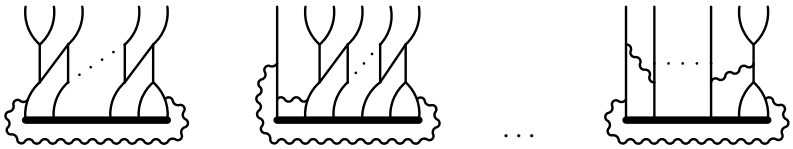


Figure 3. L-loop wrapping diagram $W_{L,0}$



(a) $W_{L,1}$ **(b)** $W_{L,2}$ **(c)** $W_{L,L-1}$

Figure 4. Sample diagrams from different classes

Now we concentrate on the wrapping diagrams. There is one wrapping graph (and its reflection) with only chiral interactions and it is depicted in figure 3. With the identification of the first and $(L+1)$-th lines we can describe its chiral structure in terms of the deformed ones and we find

$$\begin{aligned} W_{L,0} &\to (g^2N)^L K_L \left[\boldsymbol{\chi}(1, L, L-1, \ldots, 2) + \boldsymbol{\chi}(L, 1, 2, \ldots, L-1)\right] \\ &= (g^2N)^L K_L \, (q-\bar{q})^2 \left[q^{2(L-1)} + \bar{q}^{2(L-1)}\right] \ . \end{aligned} \tag{2.9}$$

The integral K_L is given in figure 2b.

Next we have all the wrapping diagrams which contain vector propagators. They can be collected in different sets distinguished by their chiral structure and correspondingly the number of vector lines that enter the graph. One representative for each set is shown in figure 4. We have:

- graphs with chiral structure $\boldsymbol{\chi}(L-1, \ldots, 2, 1)$ and 1 vector
- graphs with chiral structure $\boldsymbol{\chi}(L-2, \ldots, 2, 1)$ + 1 single Z line, and 2 vectors

JHEP08(2009)034

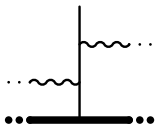
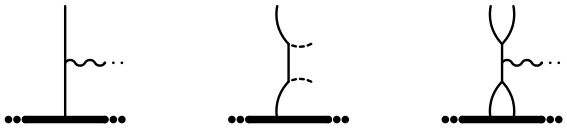


Figure 5. Z line with two V_1 vertices



(a) Block A **(b)** Block B **(c)** Block C

Figure 6. Building blocks for diagrams with vector interactions

⋮

- graphs with chiral structure $\boldsymbol{\chi}(1)$ + $(L-2)$ single Z lines, and $L-1$ vectors.

Since vectors can attach on the single Z lines via the V_1 and V_2 vertices in (2.5), the number of diagrams that one produces in this way is very, very large. Now we are going to prove that remarkable cancellations do occur. In fact we need only consider graphs with only V_2 vertices on the single Z lines since all the diagrams that contain any V_1 vertex on a single Z line do actually cancel out completely.

The proof can be organized as follows:
Let us consider a generic diagram where a single Z line has two vectors attaching to it via two V_1 vertices as shown in figure 5. This structure appears in any of the different sets of diagrams $W_{L,2}, \ldots, W_{L,L-1}$ depicted in figure 4. There are only three distinct possibilities to attach the vector on the right hand side of figure 5 to the rest of the graph, corresponding to the structures shown in figures 6a, 6b and 6c.

We examine the three situations separately and prove the complete cancellation of the divergent contributions. All the possible diagrams coming from each class are shown in figures 7, 8 and 9.

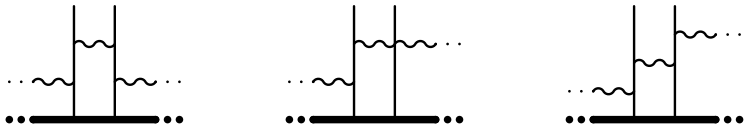
- Class A (figure 7):
  The diagram A_1 is finite.

  Performing part of the D-algebra for the diagram A_3 one immediately obtains the same structure of the diagram A_2, with a different sign due to the $\Box = -p^2$ cancelling one propagator. Since the two diagrams have the same color factor, their divergent parts sum up to zero.

- Class B (figure 8):
  For the diagrams B_1 and B_2 a partial D-algebra shows that they produce the same result while the opposite color factors make the total divergent part vanish.

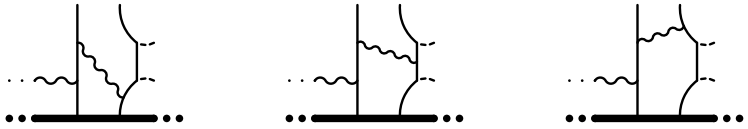
  The diagram B_3 is finite.

JHEP08(2009)034



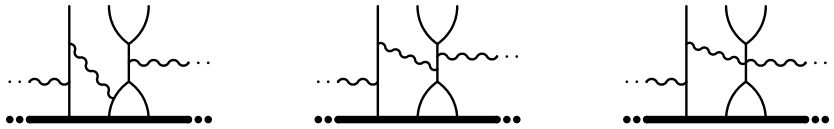
(a) A_1 **(b)** A_2 **(c)** A_3

Figure 7. Diagrams of class A



(a) B_1 **(b)** B_2 **(c)** B_3

Figure 8. Diagrams of class B



(a) C_1 **(b)** C_2 **(c)** C_3

(d) C_4 **(e)** C_5

Figure 9. Diagrams of class C

- Class C (figure 9):
  For diagrams C_1 and C_2 we have the same situation as for B_1 and B_2 and the divergent part of the two diagrams cancels out.

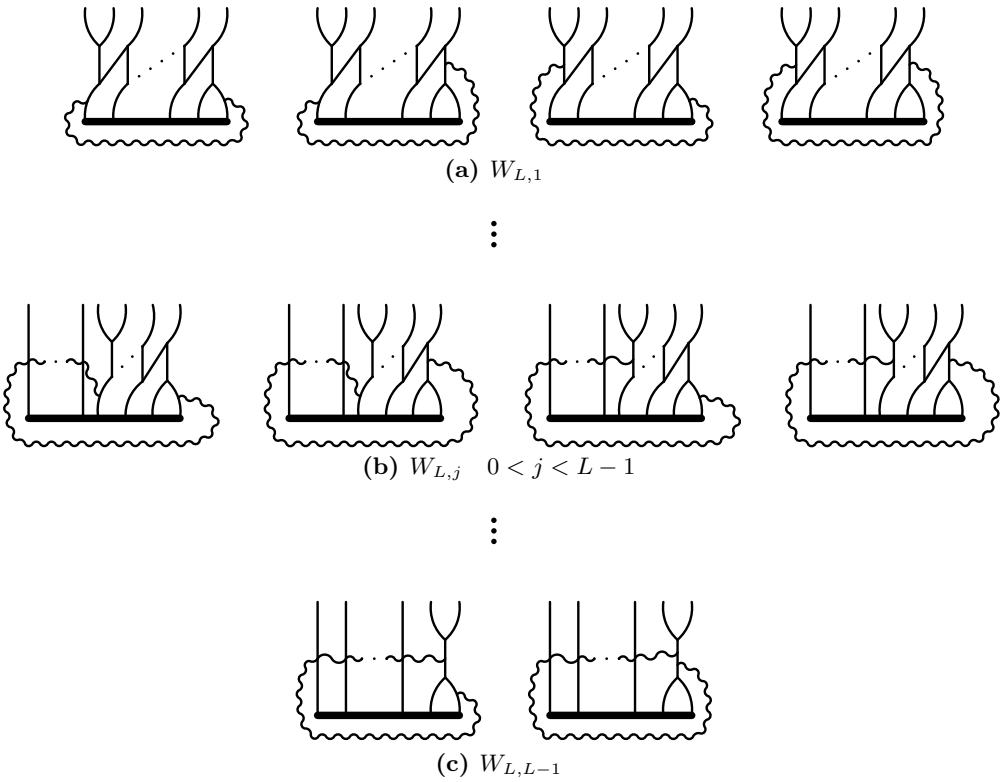
  For the diagrams C_3 and C_4 the cancellation occurs following the same pattern as for the diagrams A_2 and A_3.

  The diagram C_5 is finite.

We conclude that the only relevant diagrams with vectors are the ones in which the vectors interact with single Z lines through the V_2 vertex. In each set the number of contributing graphs is now reduced to at most four, as depicted in figure 10.

The D-algebra can be performed straightforwardly and it is easy to realize that the sum of the graphs with j vectors produces the same momentum integrals as the sum of the diagrams with $L - j - 1$ vectors.

JHEP08(2009)034



(a) $W_{L,1}$

(b) $W_{L,j} \quad 0 < j < L-1$

(c) $W_{L,L-1}$

Figure 10. Relevant diagrams after cancellations

We denote the combined color and flavor factors for the various chiral structures by $C_{L,j}$, where the subscript j stands for the number of vectors entering the diagrams. Then we have

$$C_{L,j} = (q-\bar{q})^2 \left[q^{2(L-j-1)} + \bar{q}^{2(L-j-1)}\right] = -8\sin^2(\pi\beta)\cos[2\pi\beta(L-j-1)] \tag{2.10}$$

for $j = 0, \ldots, L-1$.

Using these factors, we can write the contributions from each class:

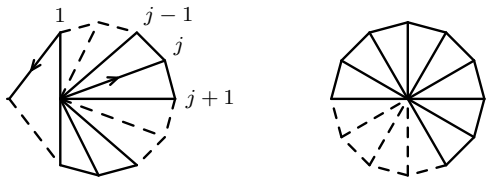
$$\begin{aligned} W_{L,0} - S_L &= (g^2N)^L \; C_{L,0}(K_L - J_L)\,, \\ &\vdots \\ W_{L,j} &= 2(g^2N)^L \; C_{L,j} I_L^{(j+1)}\,, \\ &\vdots \\ W_{L,L-1} &= -(g^2N)^L \; C_{L,L-1}(K_L - J_L)\,, \end{aligned} \tag{2.11}$$

where the integrals $I_L^{(j)}$ are shown in figure 11a.

These integrals satisfy the relation

$$I_L^{(j)} = -I_L^{(L-j+1)}\,. \tag{2.12}$$

JHEP08(2009)034



(a) $I_L^{(j)}$ **(b)** P_L

Figure 11. L-loop momentum integrals

The combination $(K_L - J_L)$, which is relevant for $(W_{L,0} - S_L)$ and for class $W_{L,L-1}$, can be written in terms of $I_L^{(1)}$ and of the integral P_L (shown in figure 11b) as

$$K_L - J_L = P_L - 2I_L^{(1)} \,. \tag{2.13}$$

We can now collect all these results and obtain the correct value for the anomalous dimension:

$$\gamma_L(\mathcal{O}_{1,L}) = \gamma_L^{\text{as}} + \delta\gamma_L(\mathcal{O}_{1,L}) \,. \tag{2.14}$$

Since both the P_L and the $I_L^{(j)}$ are free of subdivergences, their Laurent expansion in ε will present only poles of the first order. Thus we can write

$$\delta\gamma_L(\mathcal{O}_{1,L}) = -2L(g^2N)^L \lim_{\varepsilon\to 0} \varepsilon \left[(C_{L,0} - C_{L,L-1}) P_L(\varepsilon) - 2 \sum_{j=0}^{[\frac{L}{2}]-1} (C_{L,j} - C_{L,L-j-1}) I_L^{(j+1)}(\varepsilon) \right] . \tag{2.15}$$

3 Computation of the integrals

In order to obtain the actual value of the anomalous dimension for a given loop order L, we need the explicit values of the coefficients of the $1/\varepsilon$ poles in the expansions of the momentum integrals P_L and $I_L^{(j)}$.

For the integrals P_L, this result is known as a function of L [35, 36]:

$$P_L \sim \frac{1}{\varepsilon} \frac{1}{(4\pi)^{2L}} \frac{2}{L} \binom{2L-3}{L-1} \zeta(2L-3) \,, \tag{3.1}$$

where the symbol $\sim$ means that we are only interested in the divergent part.

We were not able to find a general formula for the divergent parts of the integrals $I_L^{(j)}$ as functions of L and j. However, it is possible to find recurrence relations which allow to compute the required integrals for *any* fixed value of L and j.

These recurrence relations can be found applying the technique of integration by parts, as described in [35], where integrals with the same topology, but without scalar products of momenta in the numerators, were considered. In order to follow this approach, we had to generalize the triangle rule of [35] to the case of lines with momenta in the numerators. The derivation of these generalized rules is shown in appendix A.1.

JHEP08(2009)034

Using the scalar rule (A.2), any $I_L^{(j)}$ can be written in terms of a reduced set of integrals with at most seven loops and with generic propagator weights for the lines coming out from the operator insertion. An example of this procedure is presented in appendix A.2.

The fundamental integrals can be computed explicitly using the generalized rules.

For the integrals $I_L^{(1)}$, also the technique described in [36] can be used. This allowed us to guess a general expression for $I_L^{(1)}$ as a function of L:

$$\begin{aligned} I_L^{(1)} &= \frac{1}{2}P_L + \frac{1}{L}\sum_{k=3}^{L-1}\binom{h(L,k)}{L-k}\zeta(h(L,k)+1) \\ &= \frac{1}{2}P_L + \frac{1}{L}\sum_{k=L-1-[\frac{L-1}{2}]}^{L-3}\binom{2k+1}{2k+3-L}\zeta(2k+1) + \frac{1}{2L}[1+(-1)^L](L-2)\zeta(L-1)\,, \end{aligned} \tag{3.2}$$

where $h(L,k) = 2(L-k-1+[k/2])$. We could not apply the approach of [36] to the other classes of integrals because we were not able to compute the needed higher-loop integrals with generic propagator weights. For $L \geq 10$ it also takes too long to extract the pole part from the results which were obtained from the the recurrence formula. To find the results up to $L = 11$ we have therefore employed GPXT [37] as described in appendix A.3.

Looking for a simple deformation of the formula for $I_L^{(1)}$, we were able to guess the general expression for $I_L^{(2)}$:

$$\begin{aligned} I_L^{(2)}|_{L=2m} &= \frac{1}{2}P_L - \frac{1}{L}\sum_{k=L-1-[\frac{L-1}{2}]}^{L-3}\left[\frac{2L}{L-1}(L-2-k)-1\right]\binom{2k+1}{2k+3-L}\zeta(2k+1) \\ &\quad - \frac{1}{L}(L-2)(L-1)\zeta(L-1)\,, \\ I_L^{(2)}|_{L=2m+1} &= \frac{1}{2}P_L - \frac{1}{L}\sum_{k=L-[\frac{L-1}{2}]}^{L-3}\left[\frac{2L}{L-1}(L-2-k)-1\right]\binom{2k+1}{2k+3-L}\zeta(2k+1) \\ &\quad - \frac{1}{2}(L-3)(L-1)\zeta(L)\,. \end{aligned} \tag{3.3}$$

Both these formulae have been verified up to $L = 11$.

4 Concluding remarks

We explicitly computed all the relevant $I_L^{(j)}$ up to $L = 11$. The corresponding results are shown in appendix A.4.

First of all, we see that in all the cases we considered, wrapping interactions and range-$(L+1)$ subtractions only produce transcendental contributions.

Moreover, a very precise transcendentality pattern appears, as can be seen from table 1: for every value of L, the term with maximum degree of transcendentality is always proportional to $\zeta(2L-3)$. Then a given number of terms with consecutive, lower odd degrees is present. This number is increased by one every two loops.

JHEP08(2009)034

L	Transcendental terms
$L=4$	$\zeta(3)$, $\zeta(5)$
$L=5$	$\zeta(5)$, $\zeta(7)$
$L=6$	$\zeta(5)$, $\zeta(7)$, $\zeta(9)$
$L=7$	$\zeta(7)$, $\zeta(9)$, $\zeta(11)$
$L=8$	$\zeta(7)$, $\zeta(9)$, $\zeta(11)$, $\zeta(13)$
$L=9$	$\zeta(9)$, $\zeta(11)$, $\zeta(13)$, $\zeta(15)$
$L=10$	$\zeta(9)$, $\zeta(11)$, $\zeta(13)$, $\zeta(15)$, $\zeta(17)$
$L=11$	$\zeta(11)$, $\zeta(13)$, $\zeta(15)$, $\zeta(17)$, $\zeta(19)$

Table 1. Transcendental terms produced for different values of L

This particular behaviour is also confirmed by the general expressions we guessed for a subset of the relevant classes of integrals.

Using the recurrence relations obtained from the triangle rules, one can in principle compute the exact anomalous dimension for any single-impurity, length-L operator at the critical loop order L.

Acknowledgments

This work has been supported in part by INFN, by the European Commission RTN program MRTN–CT–2004–005104 and by the Italian MIUR-PRIN contract 20075ATT78.

A Details on the computation of the integrals

A.1 Triangle rules

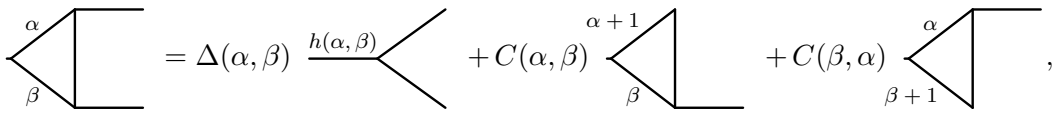
In this section we present the generalization of the triangle rule of [35] to the case of lines with momenta in the numerators, and show how they can be used to obtain recurrence relations for the integrals we need.

All the rules we need can be obtained from the following integration by parts identity, which is valid for $\alpha+\beta+1-D/2-\dim(f)/2>0$:

$$
\begin{aligned}
0 &= \int \mathrm{d}^D l \frac{\partial}{\partial l^\mu} \frac{f(l) l^\mu}{(l+p_1)^{2\alpha}(l+p_2)^{2\beta} l^2} \\
&= \int \mathrm{d}^D l \Bigg(\partial_\mu f(l) l^\mu + D f(l) \\
&\quad - 2 f(l) l^\mu \left(\alpha \frac{(l+p_1)_\mu}{(l+p_1)^2} + \beta \frac{(l+p_2)_\mu}{(l+p_2)^2} + \frac{l^\mu}{l^2} \right) \Bigg) \frac{1}{(l+p_1)^{2\alpha}(l+p_2)^{2\beta} l^2} \\
&= \int \mathrm{d}^D l \Bigg(\partial_\mu f(l) l^\mu \\
&\quad - f(l) \left(\alpha \frac{l^2 + (l+p_1)^2 - p_1^2}{(l+p1)^2} + \beta \frac{l^2 + (l+p_2)^2 - p_2^2}{(l+p_2)^2} - D + 2 \right) \Bigg) \frac{1}{(l+p_1)^{2\alpha}(l+p_2)^{2\beta} l^2}
\end{aligned}
$$

$$= \int \mathrm{d}^D l \left(\partial_\mu f(l) l^\mu - \left(\alpha \frac{l^2 - p_1^2}{(l+p_1)^2} + \beta \frac{l^2 - p_2^2}{(l+p_2)^2} + \alpha + \beta + 2 - D \right) \right) \frac{f(l)}{(l+p_1)^{2\alpha} (l+p_2)^{2\beta} l^2} \,. \tag{A.1}$$

For $f(l) = 1$ we recover the scalar rule of [35]

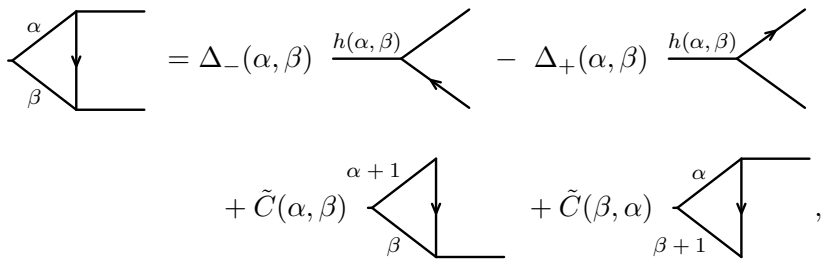


, (A.2)

where

$$\begin{aligned}
\Delta(\alpha,\beta) &= -\frac{\alpha G(\alpha+1,\beta) + \beta G(\alpha,\beta+1)}{\alpha+\beta+2-D}\,, \\
C(\alpha,\beta) &= \frac{\alpha}{\alpha+\beta+2-D}\,, \\
h(\alpha,\beta) &= \alpha+\beta+1-D/2\,, \\
G(\alpha,\beta) &= \frac{1}{(2\pi)^D} \int \frac{\mathrm{d}^D l}{l^{2\alpha}(l+p)^{2\beta}}\Big|_{p^2=1} = \frac{\Gamma(\alpha+\beta-D/2)\Gamma(D/2-\alpha)\Gamma(D/2-\beta)}{(4\pi)^{D/2}\Gamma(\alpha)\Gamma(\beta)\Gamma(D-\alpha-\beta)}\,.
\end{aligned} \tag{A.3}$$

If we take $f(l) = l^\nu$ we obtain the first generalized rule



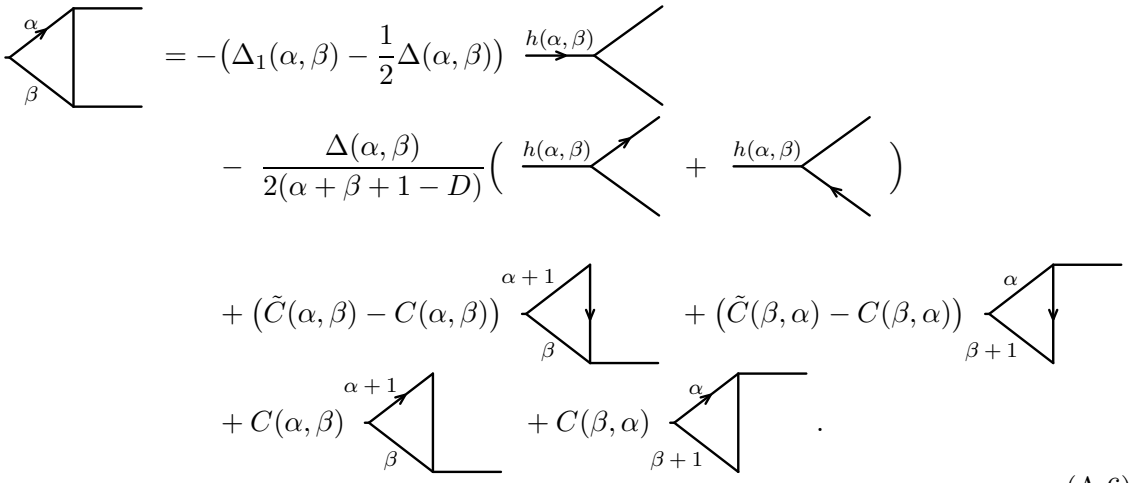
, (A.4)

where

$$\begin{aligned}
\Delta_\pm(\alpha,\beta) &= \Delta_1(\alpha,\beta) \pm \tilde{\Delta}(\alpha,\beta)\,, \\
\Delta_1(\alpha,\beta) &= \frac{(\alpha-\beta)G(\alpha,\beta) - \alpha G(\alpha+1,\beta-1) + \beta G(\alpha-1,\beta+1)}{2(\alpha+\beta+1-D)}\,, \\
\tilde{\Delta}(\alpha,\beta) &= -\frac{\alpha G(\alpha+1,\beta) + \beta G(\alpha,\beta+1)}{2(\alpha+\beta+1-D)}\,, \\
\tilde{C}(\alpha,\beta) &= \frac{\alpha}{\alpha+\beta+1-D}\,.
\end{aligned} \tag{A.5}$$

JHEP08(2009)034

Another useful formula can be derived from the previous two:



$$= -\big(\Delta_1(\alpha,\beta) - \frac{1}{2}\Delta(\alpha,\beta)\big)$$
$$- \frac{\Delta(\alpha,\beta)}{2(\alpha+\beta+1-D)}\Big($$
$$+$$
$$\Big)$$
$$+ \big(\tilde{C}(\alpha,\beta) - C(\alpha,\beta)\big)$$
$$+ \big(\tilde{C}(\beta,\alpha) - C(\beta,\alpha)\big)$$
$$+ C(\alpha,\beta)$$
$$+ C(\beta,\alpha)$$
$$. \tag{A.6}$$

A.2 Example of recurrence relation

As an example of a recurrence relation obtained from the triangle rules, let us consider the computation of $I_L^{(1)}$. We can write

$$I_L^{(1)} = K_L^{(1)}(1,\ldots,1)\,, \tag{A.7}$$

where $K_L^{(1)}(\alpha_1,\ldots,\alpha_L)$ is the integral with the same topology and numerator as $I_L^{(1)}$, but weights $\alpha_1,\ldots,\alpha_L$ on the radial lines. Using (A.2) we find

$$\begin{aligned} K_L^{(1)}(\alpha_1,\ldots,\alpha_L) &= \Delta(\alpha_{L-2},\alpha_{L-1})K_{L-1}^{(1)}(\alpha_1,\ldots,\alpha_{L-3},h(a_{L-2},\alpha_{L-1}),\alpha_L) \\ &\quad + C(\alpha_{L-2},\alpha_{L-1})G(\alpha_{L-3},\alpha_{L-2}+1)K_{L-1}^{(1)}(\alpha_1,\ldots,g(\alpha_{L-3},\alpha_{L-2}+1),\alpha_{L-1},\alpha_L) \\ &\quad + C(\alpha_{L-1},\alpha_{L-2})G(\alpha_{L-1}+1,\alpha_L)K_{L-1}^{(1)}(\alpha_1,\ldots,\alpha_{L-2},g(\alpha_{L-1}+1,\alpha_L))\,, \end{aligned} \tag{A.8}$$

where $g(\alpha,\beta) = \alpha+\beta-D/2$ and h is defined in eq. (A.5).

We can go on applying eq. (A.2) until we obtain $K_5^{(1)}$ in terms of $K_4^{(1)}$. Then, using the generalized rules, we can compute $K_4^{(1)}$ explicitly.

One can deal with integrals $I_L^{(2)}$ and $I_L^{(3)}$ in the same way, using eq. (A.2) down to five and six loops respectively. For the general case $I_L^{(j)}$ with $j \geq 4$, eq. (A.2) must be applied on both sides of the integrals to reduce them to seven loops.

A.3 GPXT in p-space

We are interested in the pole part of the integrals $I_L^{(j)}$ and P_L given in figure 11. Thereby, the relation (2.12) implies that the independent non-vanishing integrals are represented by $1 \leq j \leq [\frac{L}{2}]$, where $[x]$ denotes the integer part of x.

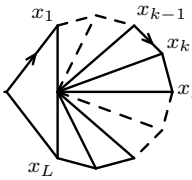
Instead of solving the integrals in x-space, we work directly in p-space. This has significant advantages: one does not have to shift the derivatives to the root vertex. Therefore,

JHEP08(2009)034

one does not encounter the problem to compute several integrals with subdivergences which combine to the required integral. Furthermore, the number of integration points is reduced by one. The calculation can therefore be pushed to higher loop order. We first write the required integral as the following linear combination

$$I_L^{(j)} = J_L^{(j)} - J_L^{(j+1)}\,, \tag{A.9}$$

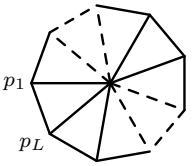
where

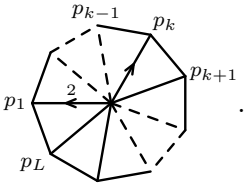
$J_L^{(k)} =$ 

$= \dfrac{1}{(2\pi)^{\mathrm{LD}}} \displaystyle\int \frac{\mathrm{d}^D p_1 \ldots \mathrm{d}^D p_L\, p_1 \cdot p_k}{p_1^4 p_2^2 \ldots p_L^2 (p_1-p_2)^2 \ldots (p_{L-1}-p_L)^2 (p_L-p_1)^2}\,.$

(A.10)

The p-space graphs of the integrals $J_L^{(k)}$ which we need for $1 \le k \le K$, $K = [\frac{L}{2}] + 1$ hence read

$J_L^{(1)} = P_L =$ 

, $\qquad J_L^{(k)} =$ 

. (A.11)

Where the number at a line denotes the weight of the propagator. We apply GPXT [37] to the above integrals in p-space. The propagators are first expanded in terms of Gegenbauer polynomials $C_i^{(1)}(x)$ (which in this case are the Chebyshev polynomials of the second kind). Then, the angular integrals are performed. This yields

$$\begin{aligned} J_L^{(1)} &= \frac{1}{(2\pi)^{\mathrm{LD}}} \frac{\Omega_{D-1}^L}{2^L} \sum_{i=0}^{\infty} \left(\frac{1}{i+1}\right)^{L-1} R_\lambda^{(1)}(i)\,, \\ J_L^{(k)} &= \frac{1}{(2\pi)^{\mathrm{LD}}} \frac{\Omega_{D-1}^L}{2^L} \frac{1}{2} \sum_{i=0}^{\infty} \sum_{\substack{i=|j-1| \\ i\neq j}}^{j+1} \left(\frac{1}{i+1}\right)^{L-k+1} \left(\frac{1}{j+1}\right)^{k-2} D_1(j,1,i) C_i^{(1)}(1) R_\lambda^{(k)}(i,j)\,, \end{aligned} \tag{A.12}$$

where Ω_{D-1} is the volume of the $(D-1)$-dimensional unit sphere, $D_1(j,1,i)$ are the Clebsch-Gordan coefficients of the Gegenbauer polynomials, and the radial integrals are given by

$$\begin{aligned} R_\lambda^{(1)}(i) &= \int_R^\infty \frac{\mathrm{d}\, r_1 \ldots \mathrm{d}\, r_L\, (r_1 \ldots\ldots r_L)^{\lambda-1}}{\max_{r_1 r_2} \ldots \max_{r_{L-1} r_L} \max_{r_L r_1}} \left(\frac{\min_{r_1 r_2}}{\max_{r_1 r_2}} \ldots \frac{\min_{r_{L-1} r_L}}{\max_{r_{L-1} r_L}} \frac{\min_{r_L r_1}}{\max_{r_L r_1}}\right)^{\frac{i}{2}}\,, \\ R_\lambda^{(k)}(i,j) &= \int_R^\infty \frac{\mathrm{d}\, r_1 \ldots \mathrm{d}\, r_L\, r_1^{\lambda-\frac{3}{2}} r_k^{\lambda-\frac{1}{2}} (r_2 \ldots r_{k-1} r_{k+1} \ldots r_L)^{\lambda-1}}{\max_{r_1 r_2} \ldots \max_{r_{L-1} r_L} \max_{r_L r_1}} \\ &\quad \times \left(\frac{\min_{r_1 r_2}}{\max_{r_1 r_2}} \ldots \frac{\min_{r_{k-1} r_k}}{\max_{r_{k-1} r_k}}\right)^{\frac{j}{2}} \left(\frac{\min_{r_k r_{k+1}}}{\max_{r_k r_{k+1}}} \ldots \frac{\min_{r_{L-1} r_L}}{\max_{r_{L-1} r_L}} \frac{\min_{r_L r_1}}{\max_{r_L r_1}}\right)^{\frac{i}{2}}\,. \end{aligned} \tag{A.13}$$

we have introduced a regulator R as a lower bound for the momentum integration. This cuts out the infrared regime of the integrals and hence does not affect the pole part we are

JHEP08(2009)034

interested in.[1] While it is simple to directly evaluate the above integrals for small L, it becomes very tedious at larger L. We found that it is advantageous to set up a recurrence relation for the radial integrals.

The starting point is the integral

$$\begin{aligned} I_1(a_1,b_1,b_2,c_1,c_2;r_0,r_2) &= \int_R^\infty \mathrm{d}\, r_1\, r_1^{a_1} \max_{r_0 r_1}^{b_1} \max_{r_1 r_2}^{b_2} \min_{r_0 r_1}^{c_1} \min_{r_1 r_2}^{c_2} \Big|_{r_0 \geq r_2} \\ &= -\frac{R^{a_1+c_1+c_2+1} r_0^{b_1} r_2^{b_2}}{a_1+c_1+c_2+1} + \frac{r_0^{b_1} r_2^{c_2}}{a_1+b_2+c_1+1} \left(\frac{(b_1-c_1) r_0^{a_1+b_2+c_1+1}}{a_1+b_1+b_2+1} + \frac{(b_2-c_2) r_2^{a_1+b_2+c_1+1}}{a_1+c_1+c_2+1} \right), \end{aligned} \tag{A.14}$$

where we assume that the constants a_1, b_1, b_2, c_1, c_2 are such that the upper boundary does not contribute. Longer chains of integration are defined as

$$\begin{aligned} &I_n(a_1,\ldots,a_n,b_1,\ldots,b_{n+1},c_1,\ldots,c_{n+1};r_0,r_{n+1}) \\ &\qquad = \int_R^\infty \mathrm{d}\, r_1 \ldots \mathrm{d}\, r_n\, r_1^{a_1} \ldots r_n^{a_n} \max_{r_0 r_1}^{b_1} \max_{r_1 r_2}^{b_2} \ldots \max_{r_n r_{n+1}}^{b_{n+1}} \\ &\qquad\qquad \min_{r_0 r_1}^{c_1} \min_{r_1 r_2}^{c_2} \ldots \min_{r_n r_{n+1}}^{c_{n+1}} \Big|_{r_0 \geq r_{n+1}} . \end{aligned} \tag{A.15}$$

It is important to remark that for the remaining domain $r_0 < r_{n+1}$ the result is obtained by simply reverting the order $(a_1,\ldots,a_n) \to (a_n,\ldots,a_1)$, $(b_1,\ldots,b_{n+1}) \to (b_{n+1},\ldots,b_1)$, $(c_1,\ldots,c_{n+1}) \to (c_{n+1},\ldots,c_1)$ and by also exchanging $r_0 \leftrightarrow r_{n+1}$.

We can then obtain a recurrence relation for the $(n+1)$-fold integral for the regime $r_0 \geq r_{n+2}$ by explicitly evaluating the integration of r_{n+1} in

$$I_{n+1}(a_1,\ldots,a_{n+1},b_1,\ldots,b_{n+2},c_1,\ldots,c_{n+2};r_0,r_{n+2}) \ .$$

If we assume again that there are no contributions from the integral boundary at infinity, guaranteed by the values of the constants a_i, b_i, c_i, and that the integrands are always polynomials, we find the following rule

$$\begin{aligned} &I_{n+1}(a_1,\ldots,a_{n+1},b_1,\ldots,b_{n+2},c_1,\ldots,c_{n+2};r_0,r_{n+2}) \\ &= \big[(r_{n+2}^{b_{n+2}} r_{n+1}^{a_{n+1}+c_{n+2}} - r_{n+1}^{a_{n+1}+b_{n+2}} r_{n+2}^{c_{n+2}}) \\ &\quad I_n(a_1,\ldots,a_n,b_1,\ldots,b_{n+1},c_1,\ldots,c_{n+1};r_0,r_{n+1})\big]_{r_{n+1}^\alpha \to \frac{1}{\alpha+1} r_{n+2}^{\alpha+1}} \\ &\quad - \big[r_{n+1}^{a_{n+1}+c_{n+2}} r_{n+2}^{b_{n+2}} I_n(a_1,\ldots,a_n,b_1,\ldots,b_{n+1},c_1,\ldots,c_{n+1};r_0,r_{n+1})\big]_{r_{n+1}^\alpha \to \frac{1}{\alpha+1} R^{\alpha+1}} \\ &\quad + \big[r_{n+1}^{a_{n+1}+b_{n+2}} r_{n+2}^{c_{n+2}} (I_n(a_1,\ldots,a_n,b_1,\ldots,b_{n+1},c_1,\ldots,c_{n+1};r_0,r_{n+1}) \\ &\qquad\qquad - I_n(a_n,\ldots,a_1,b_{n+1},\ldots,b_1,c_{n+1},\ldots,c_1;r_{n+1},r_0))\big]_{r_{n+1}^\alpha \to \frac{1}{\alpha+1} r_0^{\alpha+1}} , \end{aligned} \tag{A.16}$$

where the replacement means that one first collects all factors of r_{n+1} within each term of the corresponding expression in brackets, and then replaces the appearing factor r_{n+1} with exponent α as indicated. This mimics the integrations and can be easily evaluated with

[1] In this case, where no infrared divergences are present, one can safely set $R = 0$ as long as one keeps factors $R^{\lambda-1} = R^{-\varepsilon} \to 1$

JHEP08(2009)034

a computer. To obtain the final closed chain of integrations, we first abbreviate the chain with equal weights as

$$I_n(a,b,c;r_0,r_{n+1}) = I_n(a,\ldots,a,b,\ldots,b,c,\ldots,c;r_0,r_{n+1})\ . \tag{A.17}$$

Two open chains are then fused by identifying the respective first and second coordinate argument and then integrating over both arguments with appropriate additional power factors of these arguments. Using again the replacement rule to mimic the integrations, the combinations we need for the radial integrals are then obtained from the open chains as follows

$$\begin{aligned}
&I_L(a_1,\ldots,a_K,b_1,\ldots,b_{K-1},c_1,\ldots,c_{K-1},a,b,c)\\
&= \Big[\big[r_1^{a_1} r_K^{a_K} (I_{K-2}(a_2,\ldots,a_{K-1},b_1,\ldots,b_{K-1},c_1,\ldots,c_{K-1};r_1,r_K) I_{L-K}(a,b,c;r_1,r_K)\\
&\quad - I_{K-2}(a_{K-1},\ldots,a_2,b_{K-1},\ldots,b_1,c_{K-1},\ldots,c_1;r_K,r_1) I_{L-K}(a,b,c;r_K,r_1))\big]_{r_1^\alpha \to \frac{1}{\alpha+1} r_K^{\alpha+1}}\\
&\quad + \big[r_1^{a_1} r_K^{a_K} I_{K-2}(a_{K-1},\ldots,a_2,b_{K-1},\ldots,b_1,c_{K-1},\ldots,c_1;r_K,r_1)\\
&\qquad I_{L-K}(a,b,c;r_K,r_1)\big]_{r_1^\alpha \to \frac{1}{\alpha+1} R^{\alpha+1}}\Big]_{r_K^\alpha \to \frac{1}{\alpha+1} R^{\alpha+1}}\ .
\end{aligned} \tag{A.18}$$

The required radial integrals $R_\lambda^{(1)}(i)$ and $R_\lambda^{(k)}(i,j)$ in (A.13) are then directly given by the above expression with respectively chosen constants. For $R_\lambda^{(1)}(i)$ the constants become

$$a_1 = \cdots = a_K = a = \lambda - 1\,, \quad b_1 = \cdots = b_{K-1} = b = -1 - \frac{i}{2}\,, \quad c_1 = \cdots = c_{K-1} = c = \frac{i}{2}\,, \tag{A.19}$$

while for $R_\lambda^{(k)}(i,j)$ we take the above values except for the the following constants

$$a_1 = \lambda - \frac{3}{2}\,, \quad a_k = \lambda - \frac{1}{2}\,, \quad b_1 = \cdots = b_{k-1} = -1 - \frac{j}{2}\,, \quad c_1 = \cdots = c_{k-1} = \frac{j}{2}\ . \tag{A.20}$$

With this procedure, we could find analytic expressions for the integrals up to $L = 11$ which are listed below.

A.4 Integrals up to $L = 11$

Here we show the explicit results for the integrals $I_L^{(j)}$ which are relevant for the computation of anomalous dimensions up to $L = 11$.

$$\begin{aligned}
I_4^{(1)} &= \frac{1}{(4\pi)^8}\frac{1}{\varepsilon}\Big[\ \frac{1}{2}\,\zeta(3) + \frac{5}{2}\,\zeta(5)\ \Big]\,,\\
I_5^{(1)} &= \frac{1}{(4\pi)^{10}}\frac{1}{\varepsilon}[\ 2\,\zeta(5) + 7\,\zeta(7)]\,,\\
I_6^{(1)} &= \frac{1}{(4\pi)^{12}}\frac{1}{\varepsilon}\Big[\ \frac{2}{3}\,\zeta(5) + \frac{35}{6}\,\zeta(7) + 21\,\zeta(9)\ \Big]\,,\\
I_7^{(1)} &= \frac{1}{(4\pi)^{14}}\frac{1}{\varepsilon}[\ 3\,\zeta(7) + 18\,\zeta(9) + 66\,\zeta(11)]\,,\\
I_8^{(1)} &= \frac{1}{(4\pi)^{16}}\frac{1}{\varepsilon}\Big[\ \frac{3}{4}\,\zeta(7) + \frac{21}{2}\,\zeta(9) + \frac{231}{4}\,\zeta(11) + \frac{429}{2}\,\zeta(13)\ \Big]\,,
\end{aligned}$$

$$I_9^{(1)} = \frac{1}{(4\pi)^{18}}\frac{1}{\varepsilon}\Big[\ 4\,\zeta(9) + \frac{110}{3}\,\zeta(11) + \frac{572}{3}\,\zeta(13) + 715\,\zeta(15)\ \Big]\,,$$
$$I_{10}^{(1)} = \frac{1}{(4\pi)^{20}}\frac{1}{\varepsilon}\Big[\ \frac{4}{5}\,\zeta(9) + \frac{33}{2}\,\zeta(11) + \frac{1287}{10}\,\zeta(13) + \frac{1287}{2}\,\zeta(15) + 2431\,\zeta(17)\ \Big]\,,$$
$$I_{11}^{(1)} = \frac{1}{(4\pi)^{18}}\frac{1}{\varepsilon}[\ 5\,\zeta(11) + 65\,\zeta(13) + 455\,\zeta(15) + 2210\,\zeta(17)\ \ + 8398\,\zeta(19)]\,,$$

$$I_4^{(2)} = \frac{1}{(4\pi)^{8}}\frac{1}{\varepsilon}\Big[\ -\frac{3}{2}\,\zeta(3) + \frac{5}{2}\,\zeta(5)\ \Big]\,,$$
$$I_5^{(2)} = \frac{1}{(4\pi)^{10}}\frac{1}{\varepsilon}[\ -4\,\zeta(5) + 7\,\zeta(7)]\,,$$
$$I_6^{(2)} = \frac{1}{(4\pi)^{12}}\frac{1}{\varepsilon}\Big[\ -\frac{10}{3}\,\zeta(5) - \frac{49}{6}\,\zeta(7) + 21\,\zeta(9)\ \Big]\,,$$
$$I_7^{(2)} = \frac{1}{(4\pi)^{14}}\frac{1}{\varepsilon}[\ -12\,\zeta(7) - 24\,\zeta(9) + 66\,\zeta(11)]\,,$$
$$I_8^{(2)} = \frac{1}{(4\pi)^{16}}\frac{1}{\varepsilon}\Big[\ -\frac{21}{4}\,\zeta(7) - \frac{75}{2}\,\zeta(9) - \frac{297}{4}\,\zeta(11) + \frac{429}{2}\,\zeta(13)\ \Big]\,,$$
$$I_9^{(2)} = \frac{1}{(4\pi)^{18}}\frac{1}{\varepsilon}\Big[\ -24\,\zeta(9) - \frac{385}{3}\,\zeta(11) - \frac{715}{3}\,\zeta(13) + 715\,\zeta(15)\ \Big]\,,$$
$$I_{10}^{(2)} = \frac{1}{(4\pi)^{20}}\frac{1}{\varepsilon}\Big[\ -\frac{36}{5}\,\zeta(9) - \frac{187}{2}\,\zeta(11) - \frac{4433}{10}\,\zeta(13) - \frac{1573}{2}\,\zeta(15) + 2431\,\zeta(17)\ \Big]\,,$$
$$I_{11}^{(2)} = \frac{1}{(4\pi)^{18}}\frac{1}{\varepsilon}[\ -40\,\zeta(11) - 364\,\zeta(13) - 1547\,\zeta(15) - 2652\,\zeta(17)\ \ + 8398\,\zeta(19)]\,,$$

$$I_6^{(3)} = \frac{1}{(4\pi)^{12}}\frac{1}{\varepsilon}\Big[\ \frac{20}{3}\,\zeta(5) - \frac{14}{3}\,\zeta(7)\ \Big]\,,$$
$$I_7^{(3)} = \frac{1}{(4\pi)^{14}}\frac{1}{\varepsilon}[\ 15\,\zeta(7) - 6\,\zeta(9)]\,,$$
$$I_8^{(3)} = \frac{1}{(4\pi)^{16}}\frac{1}{\varepsilon}\Big[\ \frac{63}{4}\,\zeta(7) + \frac{81}{2}\,\zeta(9) - \frac{99}{4}\,\zeta(11)\ \Big]\,,$$
$$I_9^{(3)} = \frac{1}{(4\pi)^{18}}\frac{1}{\varepsilon}\Big[\ 56\,\zeta(9) + \frac{440}{3}\,\zeta(11) - \frac{286}{3}\,\zeta(13)\ \Big]\,,$$
$$I_{10}^{(3)} = \frac{1}{(4\pi)^{18}}\frac{1}{\varepsilon}\Big[\ \frac{144}{5}\,\zeta(9) + 209\,\zeta(11) + \frac{2431}{5}\,\zeta(13) - \frac{715}{2}\,\zeta(15)\ \Big]\,,$$
$$I_{11}^{(3)} = \frac{1}{(4\pi)^{18}}\frac{1}{\varepsilon}[\ 135\,\zeta(11) + 819\,\zeta(13) + 1638\,\zeta(15) - 1326\,\zeta(17)]\,,$$

$$I_8^{(4)} = \frac{1}{(4\pi)^{16}}\frac{1}{\varepsilon}\Big[\ -\frac{105}{4}\,\zeta(7) - \frac{15}{2}\,\zeta(9) + \frac{165}{4}\,\zeta(11)\ \Big]\,,$$
$$I_9^{(4)} = \frac{1}{(4\pi)^{18}}\frac{1}{\varepsilon}[\ -56\,\zeta(9) - 55\,\zeta(11) + 143\,\zeta(13)]\,,$$
$$I_{10}^{(4)} = \frac{1}{(4\pi)^{18}}\frac{1}{\varepsilon}\Big[\ -\frac{336}{5}\,\zeta(9) - 231\,\zeta(11) - \frac{429}{5}\,\zeta(13) + \frac{1001}{2}\,\zeta(15)\ \Big]\,,$$

JHEP08(2009)034

$$I_{11}^{(4)} = \frac{1}{(4\pi)^{18}}\frac{1}{\varepsilon}[\,-240\,\zeta(11) - 936\,\zeta(13) - 182\,\zeta(15) + 1768\,\zeta(17)]\,,$$

$$I_{10}^{(5)} = \frac{1}{(4\pi)^{18}}\frac{1}{\varepsilon}\left[\,\frac{504}{5}\,\zeta(9) + 154\,\zeta(11) - \frac{1144}{5}\,\zeta(13)\,\right],$$

$$I_{11}^{(5)} = \frac{1}{(4\pi)^{18}}\frac{1}{\varepsilon}[\,210\,\zeta(11) + 546\,\zeta(13) - 637\,\zeta(15)]\ .$$

References

[1] F. Fiamberti, A. Santambrogio, C. Sieg and D. Zanon, *Wrapping at four loops in $N = 4$ SYM*, *Phys. Lett.* **B 666** (2008) 100 [arXiv:0712.3522] [SPIRES].

[2] F. Fiamberti, A. Santambrogio, C. Sieg and D. Zanon, *Anomalous dimension with wrapping at four loops in $N = 4$ SYM*, *Nucl. Phys.* **B 805** (2008) 231 [arXiv:0806.2095] [SPIRES].

[3] F. Fiamberti, A. Santambrogio, C. Sieg and D. Zanon, *Finite-size effects in the superconformal beta-deformed $N = 4$ SYM*, *JHEP* **08** (2008) 057 [arXiv:0806.2103] [SPIRES].

[4] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [hep-th/9711200] [SPIRES].

[5] J.A. Minahan and K. Zarembo, *The Bethe-ansatz for $N = 4$ super Yang-Mills*, *JHEP* **03** (2003) 013 [hep-th/0212208] [SPIRES].

[6] N. Beisert, C. Kristjansen and M. Staudacher, *The dilatation operator of $N = 4$ super Yang-Mills theory*, *Nucl. Phys.* **B 664** (2003) 131 [hep-th/0303060] [SPIRES].

[7] N. Beisert, *The* SU(2|3) *dynamic spin chain*, *Nucl. Phys.* **B 682** (2004) 487 [hep-th/0310252] [SPIRES].

[8] M. Staudacher, *The factorized S-matrix of CFT/AdS*, *JHEP* **05** (2005) 054 [hep-th/0412188] [SPIRES].

[9] N. Beisert, V. Dippel and M. Staudacher, *A novel long range spin chain and planar $N = 4$ super Yang-Mills*, *JHEP* **07** (2004) 075 [hep-th/0405001] [SPIRES].

[10] G. Arutyunov, S. Frolov and M. Staudacher, *Bethe ansatz for quantum strings*, *JHEP* **10** (2004) 016 [hep-th/0406256] [SPIRES].

[11] R. Hernandez and E. Lopez, *Quantum corrections to the string Bethe ansatz*, *JHEP* **07** (2006) 004 [hep-th/0603204] [SPIRES].

[12] N. Beisert, R. Hernandez and E. Lopez, *A crossing-symmetric phase for $AdS_5 \times S^5$ strings*, *JHEP* **11** (2006) 070 [hep-th/0609044] [SPIRES].

[13] N. Beisert, B. Eden and M. Staudacher, *Transcendentality and crossing*, *J. Stat. Mech.* (2007) P01021 [hep-th/0610251] [SPIRES].

[14] S. Schäfer-Nameki, *Exact expressions for quantum corrections to spinning strings*, *Phys. Lett.* **B 639** (2006) 571 [hep-th/0602214] [SPIRES].

[15] S. Schäfer-Nameki, M. Zamaklar and K. Zarembo, *How accurate is the quantum string Bethe ansatz?*, *JHEP* **12** (2006) 020 [hep-th/0610250] [SPIRES].

[16] G. Arutyunov, S. Frolov and M. Zamaklar, *Finite-size effects from giant magnons*, *Nucl. Phys.* **B 778** (2007) 1 [hep-th/0606126] [SPIRES].

JHEP08(2009)034

[17] J.A. Minahan and O. Ohlsson Sax, *Finite size effects for giant magnons on physical strings*, *Nucl. Phys.* **B 801** (2008) 97 [arXiv:0801.2064] [SPIRES].

[18] M.P. Heller, R.A. Janik and T. Lukowski, *A new derivation of Lüscher F-term and fluctuations around the giant magnon*, *JHEP* **06** (2008) 036 [arXiv:0801.4463] [SPIRES].

[19] B. Ramadanovic and G.W. Semenoff, *Finite size giant magnon*, *Phys. Rev.* **D 79** (2009) 126006 [arXiv:0803.4028] [SPIRES].

[20] Y. Hatsuda and R. Suzuki, *Finite-size effects for dyonic giant magnons*, *Nucl. Phys.* **B 800** (2008) 349 [arXiv:0801.0747] [SPIRES].

[21] N. Gromov, S. Schäfer-Nameki and P. Vieira, *Quantum wrapped giant magnon*, *Phys. Rev.* **D 78** (2008) 026006 [arXiv:0801.3671] [SPIRES].

[22] O.O. Sax, *Finite size giant magnons and interactions*, *Acta Phys. Polon.* **B 39** (2008) 3143 [arXiv:0810.5236] [SPIRES].

[23] Y. Hatsuda and R. Suzuki, *Finite-size effects for multi-magnon states*, *JHEP* **09** (2008) 025 [arXiv:0807.0643] [SPIRES].

[24] C. Sieg and A. Torrielli, *Wrapping interactions and the genus expansion of the 2-point function of composite operators*, *Nucl. Phys.* **B 723** (2005) 3 [hep-th/0505071] [SPIRES].

[25] Z. Bajnok and R.A. Janik, *Four-loop perturbative Konishi from strings and finite size effects for multiparticle states*, *Nucl. Phys.* **B 807** (2009) 625 [arXiv:0807.0399] [SPIRES].

[26] V.N. Velizhanin, *The four-loop Konishi in $N = 4$ SYM*, arXiv:0808.3832 [SPIRES].

[27] R.G. Leigh and M.J. Strassler, *Exactly marginal operators and duality in four-dimensional $N = 1$ supersymmetric gauge theory*, *Nucl. Phys.* **B 447** (1995) 95 [hep-th/9503121] [SPIRES].

[28] A. Mauri, S. Penati, A. Santambrogio and D. Zanon, *Exact results in planar $N = 1$ superconformal Yang-Mills theory*, *JHEP* **11** (2005) 024 [hep-th/0507282] [SPIRES].

[29] O. Lunin and J.M. Maldacena, *Deforming field theories with $\mathrm{U}(1) \times \mathrm{U}(1)$ global symmetry and their gravity duals*, *JHEP* **05** (2005) 033 [hep-th/0502086] [SPIRES].

[30] R. Roiban, *On spin chains and field theories*, *JHEP* **09** (2004) 023 [hep-th/0312218] [SPIRES].

[31] D. Berenstein and S.A. Cherkis, *Deformations of $N = 4$ SYM and integrable spin chain models*, *Nucl. Phys.* **B 702** (2004) 49 [hep-th/0405215] [SPIRES].

[32] S.A. Frolov, R. Roiban and A.A. Tseytlin, *Gauge-string duality for superconformal deformations of $N = 4$ super Yang-Mills theory*, *JHEP* **07** (2005) 045 [hep-th/0503192] [SPIRES].

[33] D.V. Bykov and S. Frolov, *Giant magnons in TsT-transformed $AdS_5 \times S^5$*, *JHEP* **07** (2008) 071 [arXiv:0805.1070] [SPIRES].

[34] S.J. Gates Jr., M.T. Grisaru, M. Roček and W. Siegel, *Superspace, or one thousand and one lessons in supersymmetry*, *Front. Phys.* **58** (1983) 1 [hep-th/0108200] [SPIRES].

[35] D.J. Broadhurst, *Evaluation of a class of Feynman diagrams for all numbers of loops and dimensions*, *Phys. Lett.* **B 164** (1985) 356 [SPIRES].

[36] N.I. Usyukina, *Calculation of multiloop diagrams in arbitrary order*, *Phys. Lett.* **B 267** (1991) 382 [*Theor. Math. Phys.* **87** (1991) 627] [*Teor. Mat. Fiz.* **87** (1991) 414] [SPIRES].

[37] K.G. Chetyrkin, A.L. Kataev and F.V. Tkachov, *New approach to evaluation of multiloop Feynman integrals: the Gegenbauer polynomial x space technique*, *Nucl. Phys.* **B 174** (1980) 345 [SPIRES].

JHEP08(2009)034